Int. J. Heat Mass Transfer. Vol. 11, pp. 235–239. Pergamon Press 1968. Printed in Great Britain

CONVECTIVE LIQUID STABILITY IN CLOSED CIRCUITS

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(*Received* 16 *December* 1966)

Аннотация—Найдены условия возникновения конвективной циркуляции в подогреваемых снизу контурах, образованных двуя замкнутымми вертикальными круглыми трубами или щелями. Для контура из круглых труб результаты теоретического исследования подтверждены экспериментом.

NOMENCLATURE

- Ra, $= (kR)^4 = g\beta/\nu\varkappa\ AR^4$, Rayleigh number;
- g, gravitational acceleration;
- β, thermal expansion coefficient of a liquid;
- ν, kinematic viscosity coefficient of a liquid;
- $\varkappa$, thermal diffusivity of a liquid;
- R, internal duct radius or half-width of a slot;
- Bi, $= \alpha R_1/\lambda_e$, Biot number;
- α, heat-transfer coefficient at an external surface of ducts or slots;
- Bi', $= 1/Bi$;
- λ', $= \lambda_e/\lambda$, relationship between molecular heat conductivities of wall material and liquid;
- R', $= R_1/R$, ratio of external and internal radii.

STABILITY of mechanical equilibrium of liquid heated underneath (convective stability) in single vertical ducts with various cross-sections has been comparatively well studied [1–5]. Investigations into convective stability of a liquid filling the system of hydrodynamically connected channels are of no less theoretical and practical interest. Below we shall study the conditions of the onset of thermal convection in circuits consisting of two similar vertical circular ducts or slots.

1. CIRCULAR DUCTS

Consider a closed circuit filled with water and formed by two similar circular vertical ducts (Fig. 1), whose diameter and the distance between them are small as compared with their lengths. At their ends the ducts are so connected that the effect of the joints on the circulation of the liquid in the circuit may be neglected. In particular, we shall consider that hydraulic

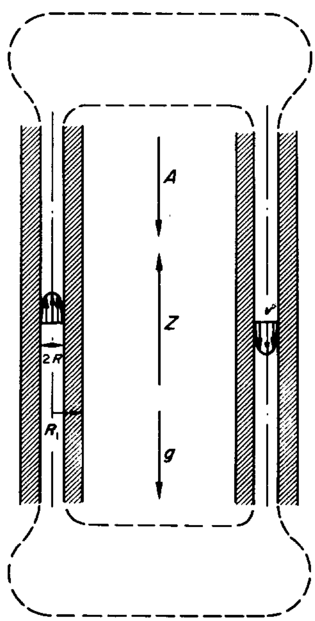

FIG. 1. Circuit diagram.

resistance of the joints is negligibly small, as compared with that of the vertical channels. Let be created in the liquid filling the vertical ducts and in their walls, a uniform downstream temperature gradient $\partial T/\partial z = -A$. Let convective heat transfer between the circuit and the surrounding medium be such that thermal interaction between the ducts through their walls may be neglected. For small values of the gradient, the liquid in the circuit will be in mechanical equilibrium; for gradients greater than a certain critical one, equilibrium will be impossible and convective motion will set.* Find the conditions of the onset of convective circulation in the circuit.

Considering only the vertical portions of the circuit and assuming convective motion velocities to be parallel walls, we write down the steady-state equations for small disturbances:

$$\nu \Delta v = -g\beta t \quad (1)$$

$$\varkappa \Delta t = -Av. \quad (2)$$

Here the disturbances of velocity $v = v_z$ and temperature t depend only upon the co-ordinates in a horizontal plane; the pressure disturbance gradient is assumed zero (no forced motion); $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. For the temperature disturbances within the wall of the ducts

$$\Delta t_e = 0. \quad (3)$$

Eliminating temperature from equations (1) and (2), we come to the equation for convection in a single vertical duct obtained by Ostroumov [1]:

$$\Delta^2 v - k^4 v = 0; \qquad k^4 \equiv \frac{g\beta}{\nu\varkappa} A. \quad (4)$$

Write down the boundary conditions:

$$v|_R = 0; \quad t|_R = t_e|_R; \quad \lambda_e \frac{\partial t_e}{\partial r}\Big|_R = \lambda \frac{\partial t}{\partial r}\Big|_R; \quad -\lambda_e \frac{\partial t_e}{\partial r}\Big|_{R_1} = \alpha t_e|_{R_1}. \quad (5)$$

The latter determines the attenuation of temperature disturbances at the external surface of the ducts due to convective heat transfer with the surroundings; here r is the distance from the axis of any duct. At $r = 0$, v and t are naturally finite, and the total liquid flow through a horizontal cross-section of a circuit is equal to zero.

Equation (4) in cylindrical co-ordinates has an axisymmetric solution of the form $J_n(kr)$ and $I_n(kr)$, in which we are interested, where J_n and I_n are the Bessel functions of real and imaginary arguments, respectively. From physical considerations it follows that the simpler form of convective circulation described by the Bessel functions with the aid of $n = 0$ will correspond to the lowest value of the critical number kR. Let us look for the solution in the form of the following linear combination J_0 and I_0:

$$v = \pm v_0 [J_0(kr) I_0(kr) - I_0(kr) J_0(kr)]. \quad (6)$$

This solution, finite in the whole region under consideration, has a maximum at $r = 0$ and tends to zero at $r = R$; v_0 is the velocity amplitude. Here and from now on the formulae with the upper sign by the amplitude [in equation (6) with a sign "plus"] describe convection in the upstream duct and with lower sign, in the downstream one.

Having substituted equation (1) for equation (6) we obtain the temperature disturbance

$$t = \pm v_0 \frac{\nu k^2}{g\beta} [J_0(kr) I_0(kR) + I_0(kr) J_0(kR)]. \quad (7)$$

The axisymmetric solution of equation (3) in the cylindrical coordinates is of the form $t_e = B + D \ln r/R$. Using equation (7) and the continuity conditions of temperature at $r = R$ and heat flux at $r = R_1$, we have:

$$t_e = B\left(1 - \frac{\ln r/R}{Bi' + \ln R'}\right);$$

$$B = \pm 2v_0 \frac{\nu k^2}{g\beta} J_0(kR) I_0(kR)$$

* Pomerantsev [6] indicated that in a similar situation convective stability may appear. The case of the channels in massif is considered [7].

and applying the continuity condition of heat fluxes at $r = R$ finally we obtain the following transcendental equation:

$$2\gamma = \left[\frac{J_1(kR)}{J_0(kR)} - \frac{I_1(kR)}{I_0(kR)}\right]kR; \quad \gamma \equiv \frac{\lambda'}{Bi' + \ln R'}. \tag{8}$$

The roots kR of this equation give the unknown critical numbers determining the onset of convective circulation. The results of the solution of equation (8) are presented in Fig. 2 (curve 1) where the values of the critical Rayleigh numbers are given as a function $\ln \gamma$. At $\gamma = 0$ (the limiting case of complete thermal insulation) the liquid in the circuit is absolutely convectively instable: the critical Rayleigh number is zero.

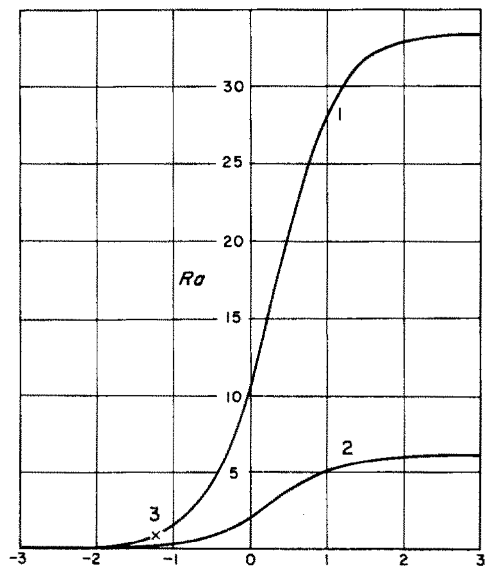

FIG. 2. Relationship between the critical Rayleigh number and γ.
(1) circular ducts,
(2) slots,
(3) experimental point.

Note that this result obtained due to thermal insulation of upstream and downstream convective flows may be directly seen from equation (2). With the rise of γ (with an increase in λ' and α, and a decrease in R') the stability curve rises steeply. At $\gamma = \infty$ (the limiting case of infinite heat conduction of the walls) the critical Rayleigh number reaches a maximum equal to 33·45.* This value is almost half the minimum critical Rayleigh number determining convective liquid stability in a single vertical duct [1]. The onset of separate antisymmetric convective motions in each duct of the circuit, without speaking about more complex separate motions, is therefore hardly probable within the above restrictions.

Note that the expression for γ is substantially simplified in the two limiting cases. At $\ln R' \gg Bi'$ (thick-wall ducts, small heat conduction of the wall material, large values of α) $\gamma \approx \lambda'/\ln R'$. In this case convective stability does not depend on heat-transfer intensity at the external duct surface. At $Bi' \gg \ln R'$ (thin-wall ducts, high heat conduction of walls, small values of α) $\gamma \approx \alpha R_1/\lambda$. Under these conditions convective stability ceases to be dependent on heat conduction of the wall material.

Experimental investigations of convective stability in circuit are carried out in the device, consisting of two glass tubes $R = 1{\cdot}54$ mm and $R_1 = 2{\cdot}2$ mm (Fig. 1) 0·75-m long, connected at the top with a large glass refrigerator vessel, cooled by the evaporation of water continuously supplied to its surface covered with a porous material. The distance between the axes of the tubes was 94 mm. The vessel and the tubes were completely filled with distilled water; the lower ends of the tubes were submerged into a heater tank of a water ultra-thermostat.

In the preliminary experiments conducted in a device, not essentially differing from the above one, the temperature field at the external surface of the vertical glass tubes was measured by a great number of thermocouples. It has been established that with laminar circulation motion of a liquid in the circuit heated at the bottom, the temperature along the vertical tubes is a

* This result may be immediately obtained from equation (7) assuming $t|_R = 0$.

linear function of the height. To determine the longitudinal temperature gradient with such a liquid motion, it appears sufficient to make thermocouple measurements only in the orifice of each of the vertical tubes. The transfer of heat from the vertical glass tubes into still air of the room (the heat-conduction coefficients necessary for the calculations were taken from the tables)*, necessary for calculating γ, was determined from the experimental temperature distribution along the tubes under non-convective conditions (when liquid was heated from above or when in each tube liquid was replaced by a metallic bar, in contact with the walls).

In Fig. 3 are presented the experimental results carried out with steady laminar convective circulation of a liquid in the circuit at different temperature drops in the heater and refrigerator. The longitudinal temperature gradient A, average along both tubes, is plotted on the abscissa, and the average transverse temperature difference $\bar{t}$ in the orifices, on the ordinate.

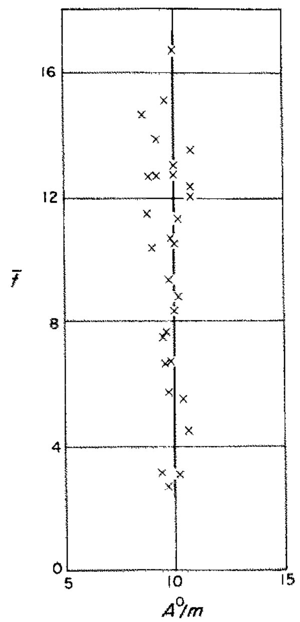

FIG. 3. Results of thermocouple measurements. Mean longitudinal temperature gradient in the circuit does not depend upon temperature difference of heater and refrigerator.

* Experiments were conducted by Fadeeva and Yarina.

It follows from the plot that A does not depend upon $\bar{t}$ within the experimental error (note that the gradients in each tube are functions of $\bar{t}$). The deviations from this trend are found only at small temperature differences of the heater and the refrigerator when, due to the decrease in convective heat transfer, linear temperature distribution may no longer occur over the whole tube length. The experiments show that $\bar{t}$ increases with the temperature drop of the heater and refrigerator and with the increased intensity of convective liquid circulation in the circuit. Conversely, in laminar circulation, the gradient A does not depend upon convective heat-transfer intensity within very wide limits (apparently, up to turbulization) and, consequently, coincides with the critical gradient which determines the onset of convective circulation.

The experimental value of the critical Rayleigh number which is found from the critical gradient for γ which in turn is calculated by using the experimental data, well coincides with the predicted one (Fig. 2).

2. SLOTS

Let us now study convective stability in a hydraulic circuit formed by two vertical slots $2R$ wide, restricted by the walls $a = R_1 - R$ thick and hydraulically connected at infinity. Considering the motion to be dependent on the co-ordinate y only and considering the earlier assumptions to be valid, we obtain the equations for small disturbances (1–4), in which now $\Delta \equiv \partial^2/\partial y^2$ (y is the co-ordinate across the slots with the origin at the centre of any of them). Boundary conditions (5) remain while r is replaced by y. The solution of equation (4) describing the simplest one-dimensional circulation motion is of the form:

$$v = \pm v_0(\cos ky \cosh kR - \cosh ky \cos kR).$$

For temperature disturbances we have

$$t = \pm v_0 \frac{\nu k^2}{g\beta}(\cos ky \cosh kR + \cosh ky \cos kR). \tag{9}$$

The solution of equation (3) is sought in the form $t_e = B + Dy$. Using equation (9) and the boundary conditions, we get at first

$$t_e = \pm 2v_0 \frac{\nu k^2}{g\beta} \frac{R_1 - y + \lambda_e/\alpha}{R_1 - R + \lambda_e/\alpha} \cos kR \cosh kR \tag{10}$$

and finally

$$2\gamma = [\tan kR - \tanh kR]\, kR; \qquad \gamma \equiv \frac{\lambda' Bi}{(Bi + 1) R' - 1}. \tag{11}$$

In the limiting cases the expression for γ is simplified. At $Bi \ll 1$ (small Biot numbers) $\gamma \approx \alpha R/\lambda$. At $Bi \gg 1$ we have $\gamma \approx R\lambda'/a$.

For the slots the stability curve is presented in Fig. 2 (curve 2). As in the case of circular ducts, at $\gamma = 0$ absolute convective instability occurs. The critical Rayleigh number increases monotonically with γ, reaching a maximum equal to 6·08 at $\gamma = \infty$.

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Abstract—Conditions are found of the onset of convective circulation in circuits heated at the bottom and formed by two closed vertical circular ducts or slots. For a circuit consisting of circular ducts, the predicted results are confirmed by experimental data.

Résumé—Les conditions du démarrage de la circulation due à la convection dans des canalisations chauffées par en bas et constituées par deux tuyaux circulaires ou plats verticaux sont obtenues. Pour la canalisation, consistant en en tuyaux circulaires, les résultats prévus sont en bon accord avec les données expérimentales.

Zusammenfassung—Es wurden die Bedingungen gefunden, für das Einsetzen einer konvektiven Umlaufströmung in Systeme, die aus zwei senkrechten Rohren mit Kreisquerschnitt oder Schlitzen gebildet sind, und von unten beheizt werden. Für das aus Rohren mit Kreisquerschnitt gebildete Umlaufsystem wurden die berechneten Ergebnisse durch experimentelle Werte bestätigt.